# FUNCTIONAL FORM FOR ESTIMATING THE LORENZ CURVE

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A flexible Lorenz curve which offers different curvatures allowed by the theory of income distribution has been introduced. The intrinsically autoregressive nature of the errors in cumulative data of the Lorenz curve is also under consideration.

Income distribution is often portrayed on a Lorenz curve. In recent years some of its functional forms have been introduced. These forms should satisfy some definitional properties, and also make estimation of the function parameters by the known estimating methods simple. This note emphasizes on two other characteristics of the Lorenz curve which have been neglected. First, Lorenz curve could be non-symetric with respect to the line $y=1-x$, for $0\leq x\leq 1$. This enables that different Lorenz curves cross the others which are the same in functional form and different in parameters for $0<x<1$ (see E.E.Hagen(2)). Figure 1 shows this phenomenon by two different Lorenz curves A and B.

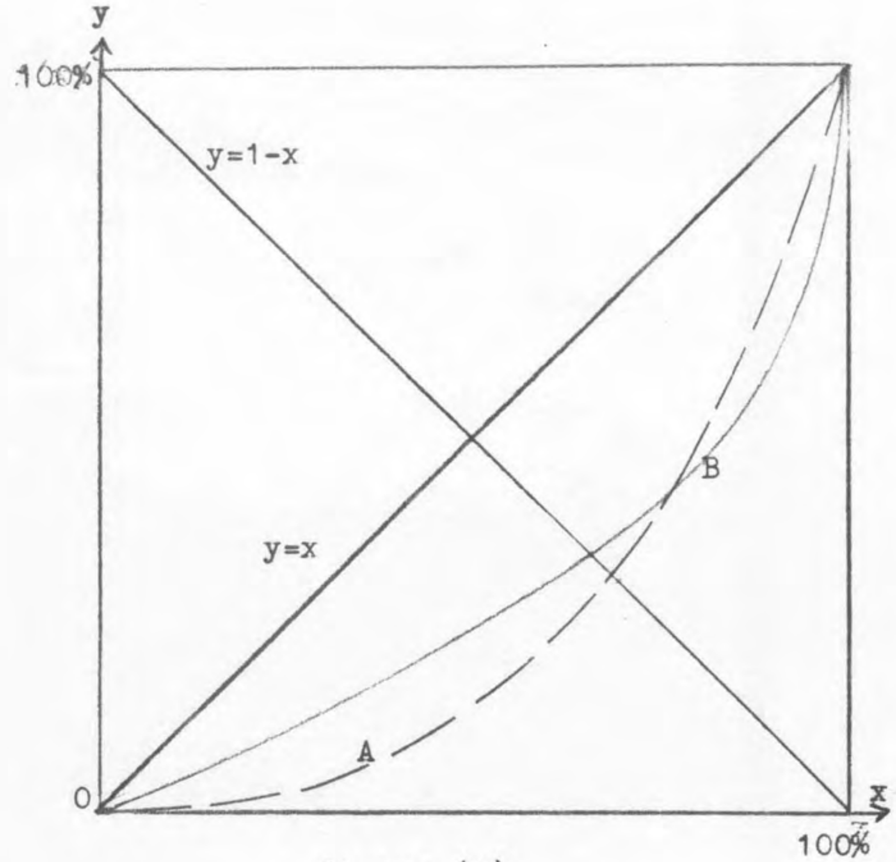


Figure (1)

M.R. Gupta(1) proposed the following definitional properties: The function $y=f(x)$ represents the Lorenz curve if:

(i) $f(0) = 0$

(ii) $f(1) = 1$

(iii) $f'(x) \geqslant 0$ for $0 \leqslant x < 1$

(iv) $f''(x) \geqslant 0$ for $0 \leqslant x \leqslant 1$

(v) $f(x) \leqslant x$ for $0 < x < 1$

(vi) $0 \leqslant \int_0^1 f(x).dx < \frac{1}{2}$

It is obvious that (vi) is redundant; when (i) to (v) are satisfied. Beacuse by manipulating (i) to (v) we have: $0 \leqslant f(x) \leqslant x$. By integrating this inequality, (vi) is derived:

$$\int_0^1 0dx \leqslant \int_0^1 f(x).dx \leqslant \int_0^1 x.dx$$

or:

$$0 \leqslant \int_0^1 f(x).dx \leqslant \tfrac{1}{2}$$

Therefore property (vi) is always satisfied, and we need no more to test (vi) for any function which satisfies (i) to (v).

Let's review the proposed functional forms:

Kakwani et. al. (4):

$M = a.N^l(2^{\frac{1}{2}}-N)^c$ where $M = \frac{x - y}{2^{\frac{1}{2}}}$ ; $N = \frac{x - y}{2^{\frac{1}{2}}}$

and $a \geqslant 0$ ; $0 \leqslant l \leqslant 1$; $0 \leqslant c \leqslant 1$ . This form does not satisfy all the properties.

Rasche et. al. (5):

$y = \left[1 - (1-x)^a\right]^{1/l}$ where $0 \leqslant a \leqslant 1$; $0 \leqslant l \leqslant 1$. This form makes estimation of the parameters by the least squares method difficult.

Gupta (1) :

$y = x.A^{x-1}$ where $A > 1$

This form satisfies definitional properties and simply can be estimated by ordinary least squares method; but by changing the parameter A (from $A_i$ to $A_j$), resulted functions ($y_i$ and $y_j$) will have never intersection for $0 < x < 1$. To prove this, we can solve the following system:

$$\begin{cases} y = x.A_i^{x-1} \\ y = x.A_j^{x-1} \end{cases}$$

Solutions are $x=y=0$ and $x=y=1$ which are not in the domain $0<x<1$.

This note suggests the following functional form which satisfies the definitional properties (i) to (v); and by changing its parameters, resulting curves may cross each others:

$$y = x^B . A^{x-1} \qquad B \geqslant 1, \quad A \geqslant 1, \quad \text{for} \quad 0 < x \leqslant 1$$

Definitional properties satisfy as follows:

(i) : $f(0)=0$.

(ii): $f(1)=1$.

(iii): $f'(x) = x^{B-1}.A^{x-1}(B + x.\log A) > 0 \quad \text{for } 0 < x < 1$

(iv): $f''(x) = x^{B-2}.A^{x-1}\left[(B + x.\log A)^2 - B\right] \geqslant 0 \quad \text{for } 0 \leqslant x \leqslant 1$

(v): $f(x) = x^B.A^{x-1} = \dfrac{x^B}{A^{1-x}} \leqslant x \quad \text{for } 0 < x < 1$

Different shapes of the function as $y = x^{B_i}.A_i^{x-1}$ and $y = x^{B_j}.A_j^{x-1}$ may have intersection for $0 < x < 1$. By solving the following system:

$$\begin{cases} y = x^{B_i}.A_i^{x-1} \\ y = x^{B_j}.A_j^{x-1} \end{cases} \qquad (1)$$

we get:

$$\frac{x-1}{\log x} = \frac{B_j - B_i}{\log(A_i/A_j)} \qquad (2)$$

It is obvious when (2) is satisfied, there is an itersection between two curves of (1). So if we solve the following system

$$\begin{cases} x-1 = B_j - B_i \\ \log x = \log(A_i/A_j) \end{cases} \qquad (3)$$

we can find a relation in terms of $A_i$, $A_j$, $B_i$ and $B_j$ which satisfies (2). Therefore:

$$A_i/A_j - 1 = B_j - B_i \qquad (4)$$

Hence, when $A_i, A_j, B_i$ and $B_j$ can satisfy equation (4), there is a solution (or intersection) for (1). But the intersection is inside the domain $0 > x > 1$ when:

By using (3) and (4)

$$\begin{cases} 0 < x = B_j - B_i + 1 < 1 \\ 0 < x = (A_i/A_j) < 1 \end{cases}$$

or:

$$\begin{cases} 0 < B_j - B_i < 1 \\ 0 < A_i < A_j \end{cases}$$

Second thing that has been ignord is the autoregressive nature of the errors in the Lorenz curve data. On the other hand, when there is an error in the (t-1)th percent of income earners, this error completely will transfer to the next cumulative percent (t). This is beacuse of using cumulative data to estimate the Lorenz curve. So if we define $u_t$ as disturbance term of the tth observation (cumulative percent), autoregressive specification of the error would be:

$$u_t = u_{t-1} + v_t$$

with $v_t$ obeying classical assumptions of regression. Therefore the stochastic form of our suggested functional form could be as follow:

$$y_t = x_t^B . A^{x_t - 1} . e^{u_t} \qquad (5)$$

or:

$$y_{t-1} = x_{t-1}^B . A^{x_{t-1} - 1} . e^{u_{t-1}} \qquad (6)$$

Dividing (5) by (6) and taking natural logarithm:

$$\mathrm{Log}(\frac{y_t}{y_{t-1}}) = B.\mathrm{Log}(\frac{x_t}{x_{t-1}}) + \mathrm{LogA}.(x_t - x_{t-1}) + u_t - u_{t-1} \qquad (7)$$

Since $u_t - u_{t-1} = v_t$ and $E(v_t, v_{t-1}) = 0$ the problem of autoregression has been discarded and (7) can be estimated by Ordinary Least Squares easily.

## REFERENCES

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